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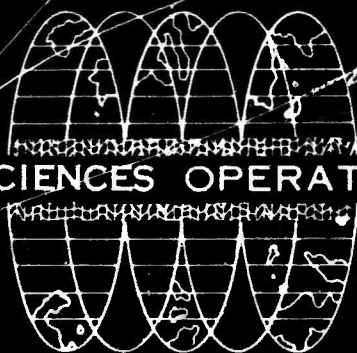
TEXAS INSTRUMENTS
INCORPORATED
SCIENCE SERVICES DIVISION

ARRAY RESEARCH PRELIMINARY REPORT
MATRIX-MULTIPLY DETECTION PROCESSING OF ARRAY DATA
SPECIAL REPORT NO. 8

By

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I. Introduction

Probabilistic processing is a method of processing the output
data of an array of seismometers with the aim of detecting earthquake
or underground nuclear blast signals in the presence of ambient seismic
noise. This method is based on the assumption that the array output
is Gaussian with mean zero and known covariance matrix Ω_1 or Ω_2, depending
on the absence or presence of signal.

The decision regarding presence or absence of the signal is, therefore,
made equivalent to testing the hypothesis that the observed data is from
a Gaussian population with covariance matrix Ω_1 with the simple alternative
hypothesis that the covariance matrix is Ω_2.

The univariate situation is: a single value x is observed from a
Gaussian population with mean zero. It must be decided whether x is
more representative of a Gaussian population with variance σ_1^2 or σ_2^2 .
The two density functions are

$$f_1(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \quad \exp\,(-x^2/2\sigma_1^2)$$

$$(1)$$

$$f_2(x) = \frac{1}{\sqrt{2\pi}\,\sigma_2} \quad \exp\,(-x^2/2\sigma_2^2)$$

A. B.
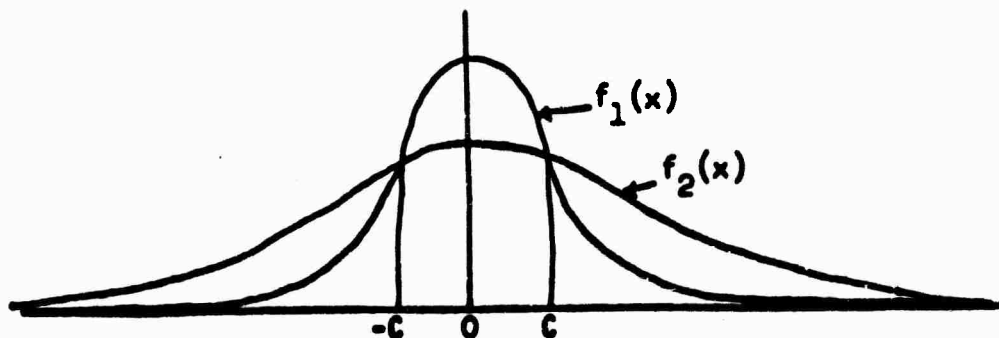
where $\sigma_2^2 > \sigma_1^2$. Clearly it is more likely that x is from $f_1(x)$ if $|x| < c$ and more likely that x is from $f_2(x)$ if $|x| > c$. So the method of processing in this case reduces to just squaring the observation x and comparing the result with a fixed constant.

If a vector x of dimension k is observed, then the two density functions become:

$$f_1(x) = (2\pi)^{-k/2} |\Omega_1|^{-1/2} \exp(-x^T \Omega_1^{-1} x/2)$$

$$f_2(x) = (2\pi)^{-k/2} |\Omega_2|^{-1/2} \exp(-x^T \Omega_2^{-1} x/2).$$ (2)

These equations can be compared to find the set of vectors x for which $f_1(x) > f_2(x)$. Thus if

$$x^T(\Omega_1^{-1} - \Omega_2^{-1})x \geq c = \ln(|\Omega_2|/|\Omega_1|)$$ (3)

then $f_2(x) \geq f_1(x)$.

The test statistic $x^T(\Omega_1^{-1} - \Omega_2^{-1})x$ must, therefore, be computed from the data. The detailed calculation of the test statistic in terms of array data is given in the following section.

A more rigorous development of the test statistic based on Bayes Theorem has been presented in a preceeding special report.[1]

II. Calculation of the Test Statistic

A. Notation

The data from an array of seismometers can be represented by the matrix X where the rows correspond to the seismometers and the columns

to the time sampled data

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & & & \\ x_{M1} & x_{M2} & \cdots & x_{MN} \end{bmatrix} .$$

Thus, x_{ij} is the observation from seismometer i at the time j.

The elements of the matrix X can be "strung out" into a single vector y in the following manner:

$$y^T = (x_1^T, x_2^T, \ldots x_N^T)$$

where $x_j^T = (x_{1j}, x_{2j}, \ldots, x_{Mj})$. This vector y is now the vector of observations from which the test statistic (equation 3) will be computed. Since the mean of y is assumed to be zero, the covariance matrix of y is just

$$\Omega_y = \overline{yy^T}$$

and the dimension of Ω_y is MN by MN since there are MN elements in the vector y. Let $r_{|i-j|+1} = \overline{x_i x_j^T}$ if $j \geq i$ and $r_{|i-j|+1}^T = \overline{x_i x_j^T}$ if $j < i$ and write

$$\overline{yy^T} = \begin{bmatrix} x_1 x_1^T & x_1 x_2^T & x_1 x_3^T & \cdots & x_1 x_N^T \\ x_2 x_1^T & x_2 x_2^T & x_2 x_3^T & \cdots & x_2 x_N^T \\ \vdots & & & & \\ x_N x_1^T & \cdots & & & x_N x_N^T \end{bmatrix} \qquad (4)$$

so that

$$\Omega_y = \begin{bmatrix} r_1 & r_2 & \cdots & r_N \\ r_2^T & r_1 & \cdots & r_{N-1} \\ \vdots & & & \\ r_N^T & \cdots & & r_1 \end{bmatrix} = R^N$$

where each submatrix r_i is an M by M matrix.

B. Inverse of R^N

The inverse of R^N can be obtained from the solutions of the two systems of equations

$$R^N \begin{pmatrix} I \\ \Gamma_2^N \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_N^N \end{pmatrix} = \begin{pmatrix} P_N \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \text{and} \quad R^N \begin{pmatrix} \Gamma_N^{N'} \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_2^{N'} \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ P_N' \end{pmatrix} . \quad (5)$$

Partition R^N in the fashion

$$R^N = \left[\begin{array}{c|c} R^{N-1} & B \\ \hline B^T & D \end{array} \right]$$

then

$$(R^N)^{-1} = \left[\begin{array}{c|c} A & \cdot \\ \hline \cdot & \cdot \end{array} \right] + \left[\begin{array}{c|c} AB\delta_1^{-1}B^TA & -AB\delta_1^{-1} \\ \hline -\delta_1^{-1}B^TA & \delta_1^{-1} \end{array} \right]$$

where the dot indicates a matrix of zeros and

$$A = (R^{N-1})^{-1}$$

$$\delta_1 = D - B^TAB.$$

From equation (5) the last M columns of $(R^N)^{-1}$ are

$$\begin{bmatrix} \Gamma_N^{N'}(P_N')^{-1} \\ \Gamma_{N-1}^{N'}(P_N')^{-1} \\ \cdot \\ \cdot \\ \Gamma_2^{N'}(P_N')^{-1} \\ (P_N')^{-1} \end{bmatrix}$$

so that

$$\begin{pmatrix} \Gamma_N^{N'}(P_N')^{-1} \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_2^{N'}(P_N')^{-1} \\ (P_N')^{-1} \end{pmatrix} = \begin{pmatrix} -AB\delta_1^{-1} \\ \\ \delta_1^{-1} \end{pmatrix} .$$

Since $(P_N')^{-1} = \delta_1^{-1}$ and $\begin{Bmatrix} \Gamma_N^{N'} \\ \cdot \\ \cdot \\ \Gamma_2^{N'} \end{Bmatrix} = -AB$, it follows that

$$AB\delta_1^{-1}B^TA = \begin{pmatrix} \Gamma_N^{N'} \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_2^{N'} \end{pmatrix} (P_N')^{-1} \begin{pmatrix} \Gamma_N^{N'} \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_2^{N'} \end{pmatrix}^T \qquad (6)$$

Now partition

$$R^N = \begin{bmatrix} D & C^T \\ \hline C & R^{N-1} \end{bmatrix}$$

to get

$$(R^N)^{-1} = \begin{bmatrix} \cdot & \vdots & \\ \hline \cdot & \vdots & A \end{bmatrix} + \begin{bmatrix} \delta_2^{-1} & \vdots & -\delta_2^{-1}C^TA \\ \hline -AC\delta_2^{-1} & \vdots & AC\delta_2^{-1}C^TA \end{bmatrix}$$

where $\delta_2^{-1} = D - C^TAC$. As before, it follows that

$$AC\delta_2^{-1}C^TA = \begin{pmatrix} \Gamma_2^N \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_N^N \end{pmatrix} (P_N)^{-1} \begin{pmatrix} \Gamma_2^N \\ \cdot \\ \cdot \\ \cdot \\ \Gamma_N^N \end{pmatrix}^T . \qquad (7)$$

Denote by s_{ij}^N the ijth M by M submatrix of the inverse of R^N. From (6) and (7)

the following equations can be written:

$$s_{ij}^{N} = s_{ij}^{N-1} + \Gamma_{N-i+1}^{N'} (P_N')^{-1} \Gamma_{N-j+1}^{N' \ T} \tag{8}$$

$$i, j = 1, \ldots, N-1$$

$$s_{i+1, \ j+1}^{N} = s_{ij}^{N-1} + \Gamma_{i+1}^{N} (P_N)^{-1} \Gamma_{j+1}^{N \ T} \tag{9}$$

$$i, j = 1, \ldots, N-1$$

Now by subtracting (8) from (9), the basic iterative formula of equation (10) is obtained

$$s_{i+., \ j+1}^{N} = s_{ij}^{N} + \Gamma_{i+1}^{N}(P_N)^{-1} \Gamma_{j+1}^{N \ T} - \Gamma_{N-i+1}^{N'}(P_N')^{-1} \Gamma_{N-j+1}^{N' \ T} \tag{10}$$

and the entire inverse matrix can be generated starting with the first M columns of the inverse which is the first column of the s_{ij}^{N}'s available from equation (5).

C. The Quadratic Processor

Suppose the matrix $X = (x_1, \ldots, x_T)$ is observed. Then the test statistic

$$P(y_i) = y_i C y_i^{T}$$

can be evaluated for $i = 1, \ldots, T-N+1$ where

$$y_i^{T} = (x_i^{T}, \ldots, x_{N+i-1}^{T})$$
$$C = \Omega_1^{-1} - \Omega_2^{-1}$$

A program to perform this calculation has been written. In addition, an integrated, squared multichannel Wiener filter output is available as an option in the quadratic processor program to give a comparison between the two detection methods.

D. Multichannel Time Series Data Generation

It is desired to generate multichannel data $x_i^T = (x_{1i}, x_{2i}, \ldots, x_{Mi})$ with the covariance matrix R^N. Let Γ_i^N $i = 2, \ldots, N$ and P_N satisfy the system of equations (5). Suppose y_i^T is a supply of vectors satisfying

$$Ey_i = 0$$

$$Ey_i y_i^T = I.$$

Then x_i will be generated by the following equation[**]

$$x_i = -(\Gamma_2^N)^T x_{i-1} - \cdots - (\Gamma_N^N)^T x_{i-N} + \epsilon_i \qquad (11)$$

for $i \geq N$ where $x_1 = \epsilon_1, x_2 = \epsilon_2, \ldots, x_{N-1} = \epsilon_{N-1}$. The method of determining the ϵ_i is

$$\epsilon_i = Hy_i$$

where $HH^T = P_N$. The reason for this is that ϵ_i must satisfy

$$E\epsilon_i \epsilon_i^T = P_N.$$

Determination of an H satisfying the above is accomplished by requiring that H be lower triangular.

The x data generated by Equation 11 has the correlation matrix R^N of Equation 5. This is established by multiplying Equation 11 on the right by x_{t-i}^T and then taking the expected value.

[**] This equation was given incorrectly in Section IV of the Array Research Semiannual Report No. 2.

E. Eigenvalues

The test statistic is

$$P(x) = x^T (\Omega_1^{-1} - \Omega_2^{-1})x \tag{12}$$

where Ω_1 and Ω_2 are Toeplitz matrices. Let the observation vector x be transformed to z by the NM by NM nonsingular transformation $x = Sz$ such that

$$P(x) = z^T S^T (\Omega_1^{-1} - \Omega_2^{-1}) Sz = z^T Dz \tag{13}$$

where D is diagonal. Then the computation of the quadratic form is reduced to filtering operations, followed by a summation of the squared outputs. A especially important transformation, diagonalizing both Ω_1^{-1}, Ω_2^{-1} simultaneously, exists since $A = \Omega_1^{-1}$ and $B = \Omega_2^{-1}$ are positive definite.[2] This transformation is found by solving the following generalized eigenvalue problem.

The solution for λ's satisfying

$$|A - \lambda B| = 0 \tag{14}$$

is called the generalized eigenvalue problem. The corresponding set of vectors x such that

$$Ax = \lambda Bx \tag{15}$$

are the generalized eigenvectors. The matrix M whose columns consist of these eigenvectors normalized so that $x^T Bx = 1$ can then be shown to satisfy

$$M^T AM = \Lambda$$
$$M^T BM = I \tag{16}$$

where Λ is a diagonal matrix with elements λ.

Thus if the S of equation (13) is taken to be M, then the diagonal matrix D satisfies

$$D = \Lambda - I. \tag{17}$$

III. Theoretical Evaluation

A. Quadratic Processor

It is desirable to determine operating parameters for the quadratic detection function

$$P(x) = x^T (\Omega_1^{-1} - \Omega_2^{-1})x. \tag{18}$$

Two useful parameters are the false alarm rate α and the failure to detect rate β. These parameters are defined in terms of the distribution of $P(x)$ when x is from Ω_1, say f_I, and when x is from Ω_2, say f_{II}. The covariance matrix Ω_1 will represent noise and Ω_2 will represent signal plus noise. The parameters α, β can now be expressed by

$$\alpha = \text{Prob} \left\{ P(x) > c \mid x \in I \right\}$$
$$\beta = \text{Prob} \left\{ P(x) < c \mid x \in II \right\} \tag{19}$$

and are the probabilities of the two types of error which can be made. The other situations

$$P(x) < c \mid x \in I$$
$$P(x) > c \mid x \in II \tag{20}$$

represent correct decisions by the method of probabilistic processing.

Suppose the data x is transformed by the matrix M of section E. Then

$$x = My$$
$$P(x) = y^T M^T (\Omega_1^{-1} - \Omega_2^{-1})My$$
$$= y^T (\Lambda - I)y \tag{21}$$

where Λ is a diagonal matrix with elements λ_i satisfying the generalized eigenvalue equation

$$\left| \Omega_1^{-1} - \Omega_2^{-1} \lambda \right| = 0. \tag{22}$$

It follows from the definitions of Ω_1 and Ω_2 that $\lambda > 1$. Another equivalent expression for 22 is

$$\left| \Omega_2 \Omega_1^{-1} - \lambda I \right| = 0 \tag{23}$$

so that

$$\left| \Omega_2 \Omega_1^{-1} \right| = \left| \Omega_1^{-1} \Omega_2 \right| = \prod_i \lambda_i . \tag{24}$$

Thus the constant c of Equation 3 can be written

$$c = \ln \prod_i \lambda_i = \sum_i \ln \lambda_i . \tag{25}$$

If we insert $\Omega_1 = N$ and $\Omega_2 = S + N$ into Equation 23 it follows that

$$\left| SN^{-1} - (\lambda - 1) I \right| = 0. \tag{26}$$

Since S and N are positive definite, the roots of SN^{-1} are positive and therefore $\lambda_i > 1$.

The distribution of y, if $x \in II$, is multivariate normal with mean zero and covariance matrix $M^{-1T} \Omega_2 M^{-1} = I$. It follows from Equation 21 that $P(x)$ has the same distribution as

$$\sum_{i=1}^{P} \eta_i \chi^2_{1(i)} \tag{27}$$

where $\eta_i = \lambda_i - 1$ and the $\chi^2_{1(i)}$ are independent chi-square random variables each with 1 degree of freedom.

If $x \in I$, then y is multivariate normal with covariance $M^{-1} \Omega_1 M^{-1T} = \Lambda^{-1}$. Now transform y to a by

$$y = \Lambda^{-1/2} a \tag{28}$$

so that covariance of a is I and

$$P(x) = y^T (\Lambda - I)y = a^T \Lambda^{-1/2} (\Lambda - I)\Lambda^{-1/2} a$$
$$= a^T (I - \Lambda^{-1})a \tag{29}$$

so that $P(x)$ again has the distribution as the expression (27) with

$$\eta_i = 1 - \lambda_i^{-1} \tag{30}$$

This result could also be established by defining the columns of M' to be the eigen vectors corresponding to the eigenvalues λ' satisfying

$$\left| \Omega_2^{-1} - \lambda' \Omega_1^{-1} \right| = 0 \ ,$$

where

$$M'^T \Omega_1^{-1} M' = I$$

$$M'^T \Omega_2^{-1} M' = \Lambda'$$

and if $\qquad x = M'y$, then

$P(x) = y^T (I - \Lambda')y$. Now, when $x \in I$, y is multivariate normal with mean zero and covariance matrix I so that $P(x)$ is distributed as Equation 27 with $\eta_1 = 1 - \lambda'$ but $\lambda' = 1/\lambda$ yielding the same result as before.

The exact distribution of Equation 27 is not known in closed form. Several approximations have been considered in the literature.[3]

We will be content to assume that Equation 27 is approximately distributed as $\delta\chi_v^2$ where δ, v are determined by setting the mean and variance of $\delta\chi_v^2$ equal the mean and variance of Equation 27. This results in

$$\delta = \sigma^2/2\mu$$

$$v = 2\mu^2/\sigma^2 \tag{31}$$

where

$$\mu = \sum_{i=1}^{P} \eta_i$$

$$\sigma^2 = 2\sum_{i=1}^{P} \eta_i^2 \tag{32}$$

It is possible to arrive at the above approximation by applying a result of Patnaik[4], which relates a non-central chi-square variable to a constant multiple of a chi-square

$$\chi_{\eta,\tau}'^2 \simeq \delta\chi_v^2 \quad , \tag{33}$$

to each term of Equation 27; then, use the addition theorem for non-central chi-square values, and finally apply Patnaik's result to the summed non-central chi-squares. The advantage of the above approach is in transmitting the known qualities of the approximation (33) to the approximation given by Equations 31 and 32.

The above results can be applied to the pair of Equations (19) to obtain

$$\alpha = \text{Prob} \left\{ \chi_v^2 > c/\delta \mid \eta_i = 1 - \lambda_i^{-1} \right\}$$

$$\beta = \text{Prob} \left\{ \chi_v^2 < c/\delta \mid \eta_i = \lambda_i - 1 \right\} \tag{34}$$

where v and δ are defined in terms of the η_i values by Equations 31 and 32.

Note that both the approximate and true means and variances are given by

$$\mu_I = \sum_{i=1}^{P} (1 - \lambda_i')$$

$$\mu_{II} = \sum_{i=1}^{P} (\lambda_i - 1)$$

$$\sigma_I^2 = 2 \sum_{i=1}^{P} (1 - \lambda_i')^2 \tag{35}$$

$$\sigma_{II}^2 = 2 \sum_{i=1}^{P} (\lambda_i - 1)^2$$

and that since each term of μ_I is less than the corresponding term of μ_{II} it follows that $\mu_{II} > \mu_I$, $\sigma_{II}^2 > \sigma_I^2$ as expected.

It is not necessary to calculate the individual λ's to apply the approximation or to obtain the parameters given by Equations 35. From section E,

$$\Omega_1^{-1} M = \Omega_2^{-1} M \Lambda \tag{36}$$

so that

$$M^{-1} \Omega_2 \Omega_1^{-1} M = \Lambda \tag{37}$$

and

$$\sum \lambda_i = \text{tr } \Lambda = \text{tr } \Omega_2 \, \Omega_1^{-1} \quad . \tag{38}$$

Also

$$\sum \lambda_i^2 = \text{tr } \Lambda\Lambda = \text{tr } M^{-1} \, \Omega_2 \, \Omega_1^{-1} \, \Omega_2 \, \Omega_1^{-1} \, M$$

$$= \text{tr } \Omega_2 \, \Omega_1^{-1} \, \Omega_2 \, \Omega_1^{-1} \tag{39}$$

and it follows that

$$\sum \lambda_i' = \text{tr } \Omega_1 \, \Omega_2^{-1}$$

$$\sum \lambda_i'^2 = \text{tr } \Omega_1 \, \Omega_2^{-1} \, \Omega_1 \, \Omega_2^{-1} \quad . \tag{40}$$

Thus the approximation parameters v, δ, and the population means and variances can be written in terms of the original correlation matrices and their inverses.

B. Squared Multichannel Wiener Filter

The optimum Wiener coefficients to estimate the signal at a selected seismometer are given by

$$\Omega_2 \, W = \gamma \tag{41}$$

where

Ω_2 = covariance matrix of signal plus noise (NM by NM)

W = optimum filter coefficients (NM by 1)

γ = selected signal correlations (NM by 1).

The original array data is then filtered

$$y_t = \sum_{i=1}^{N} V^T \cdot_{t-i} = W^T x \tag{42}$$

where

$$W = \begin{pmatrix} W_1 \\ W_2 \\ \\ W_N \end{pmatrix}$$

and the resulting time trace y_t is the optimum (mean square error) estimate of the signal trace. It is desirable to consider the output squared y_t^2, as a competitor of $P(x)$ for signal detection. We therefore want the istribution of y_t^2 when $x_t \in I$ and when $x_t \in II$. Since y_t is a linear combination of normal random variables, y_t is normal with mean zero and variance

$$\sigma^2 = W^T \Omega_x W. \tag{43}$$

Thus y_t^2 has the same distribution as $\sigma^2 \chi_1^2$. Since $W = \Omega_2^{-1} \gamma$, σ^2 can be written

$$\sigma^2 = \gamma^T \Omega_2^{-1} \Omega_x \Omega_2^{-1} \gamma \ . \tag{44}$$

If $x \in II$, then $\Omega_x = \Omega_2$ and

$$\sigma^2 = \gamma^T \Omega_2^{-1} \gamma = \gamma^T W \tag{45}$$

whereas

$$\sigma^2 = \gamma^T \Omega_2^{-1} \Omega_1 \Omega_2^{-1} \gamma \tag{46}$$

when $x \in I$. The quantity $\gamma^T W$ is simply related to the mean square error which is output in the multichannel filter program. which determines the filter coefficients W.

IV. Discussion of Processing Results

Correlation matrices for an annular ring noise model (3-4 km/sec) and a solid disk (8 km/sec) signal model were generated by an existing program[5] and used as input data to the sequence of programs discussed in Section II. The frequency spectrum for both signal and noise was left white. The center and three outer seismometer locations of the triangular WMSO array were used thereby generating 4 by 4 correlation submatrices for 25 lags.
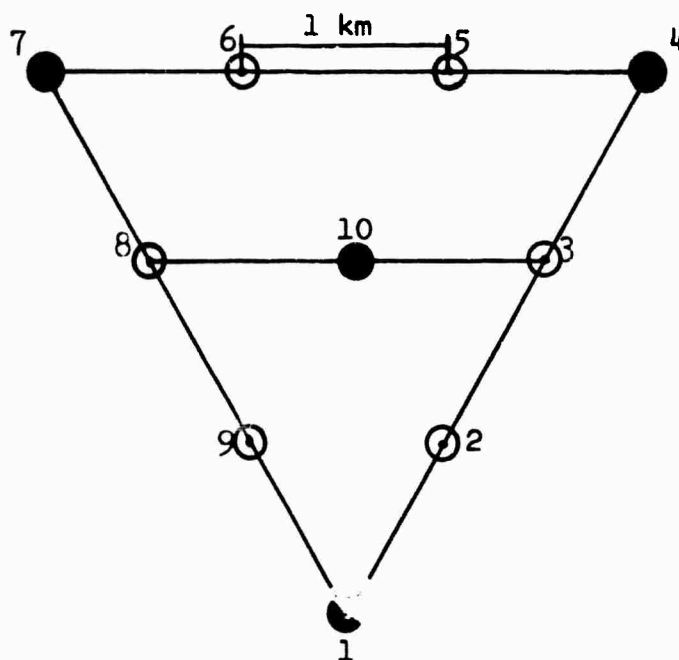
Geometry of the WMSO Array Locations Used.

The 10 lag matrices for both noise and signal plus noise were inverted and the P,Γ matrices for each were written on CPT to be used in the data generation program. The random error values c_t are different (independently generated) for each of the 6 traces shown in Figures 1 and 2. The difference of the 2 inverses was also written on CPT to be used in the program which evaluates the quadratic processor. A plot of the quadratic processor output as a function of

time is given in figure 1. The left and right portions are quadratic processor output for data generated with the noise correlation matrix and signal plus noise correlation matrix respectively. The horizontal line separating the two curves is drawn at the amplitude c of Section II. The mean and variance (average sum of squares of deviations about the mean) calculated from the two plots of figure 1 and corresponding theoretical values are

$$m_1 = 16.0 \qquad \mu_I = 17.0$$

$$s_1^2 = 11.8 \qquad \sigma_I^2 = 18.0$$

$$m_2 = 44.4 \qquad \mu_{II} = 44.1$$

$$s_2^2 = 217.4 \qquad \sigma_{II}^2 = 227.4$$

The values of μ and σ^2 very accurately predict the calculated values m and s^2. These results show that the effectiveness of the quadratic processor may be conveniently determined from the correlation matrices on a theoretical basis.

There are 4 points of the 480 noise points greater than c_{10} and 8 points of the 440 signal plus noise points less than c_{10} (ignoring the first 20 points of the second trace due to an unfortunate end effect). The measured values of α, β are approximately .008 and .018 respectively. The theoretical values of α, β are given by

$$\alpha = \text{Prob} \left\{ x_{32}^2 > 50 \right\} = .014$$

$$\beta = \text{Prob} \left\{ x_{17}^2 < 10 \right\} < .14 .$$

Exact values are unavailable for α, β due to inadequate x^2 tables.

The same models, using 25 lags, were also evaluated and the results are summarized in Table 1. The quadratic processor output is also plotted in Figure 1.

The theoretical values for α, β for 25 lags are approximately

$$\alpha = \text{Prob } \chi^2_{80} > 120 = .0013$$

$$\beta = \text{Prob } \chi^2_{44} < 20.4 = .005$$

where the normal approximation has been used.

From the evaluation of Equations 45 and 46, it was found that for 25 lags the squared four channel Wiener filter is distributed as

$$.65 \, \chi^2_1 \quad \text{if} \quad x \in \text{II}$$

$$.18 \, \chi^2_1 \quad \text{if} \quad x \in \text{I.}$$

The filters were determined and applied to data generated from populations I and II. These results are plotted in Figure 2. The sample means of I and II are .72 and .19.

No natural critical level c is available as for the quadratic detector. For comparison suppose we consider $\alpha = \beta$. Then the critical level is approximately .15 and $\alpha = \beta = .36$.

It should be noted in the above comparison that no summation over the squared output has been made. The reason for this is that $(25) \cdot (4)$ of the original data points will produce one output point by either of the two detection methods. If one sums over a gate length other than 1 for the Wiener trace, then more data is used and it would only be fair to allow smoothing for the quadratic processor.

One argument against the above is that summing for one of the two methods may be more beneficial than for the other. The "whiteness" of the Wiener outputs suggests smoothing is of greater importance to the Wiener process than to the quadratic process. The results of Figures 1 and 2 for 25 lags were smoothed over a gate length of 10 and 25 points and the results shown in Figures 3 and 4.
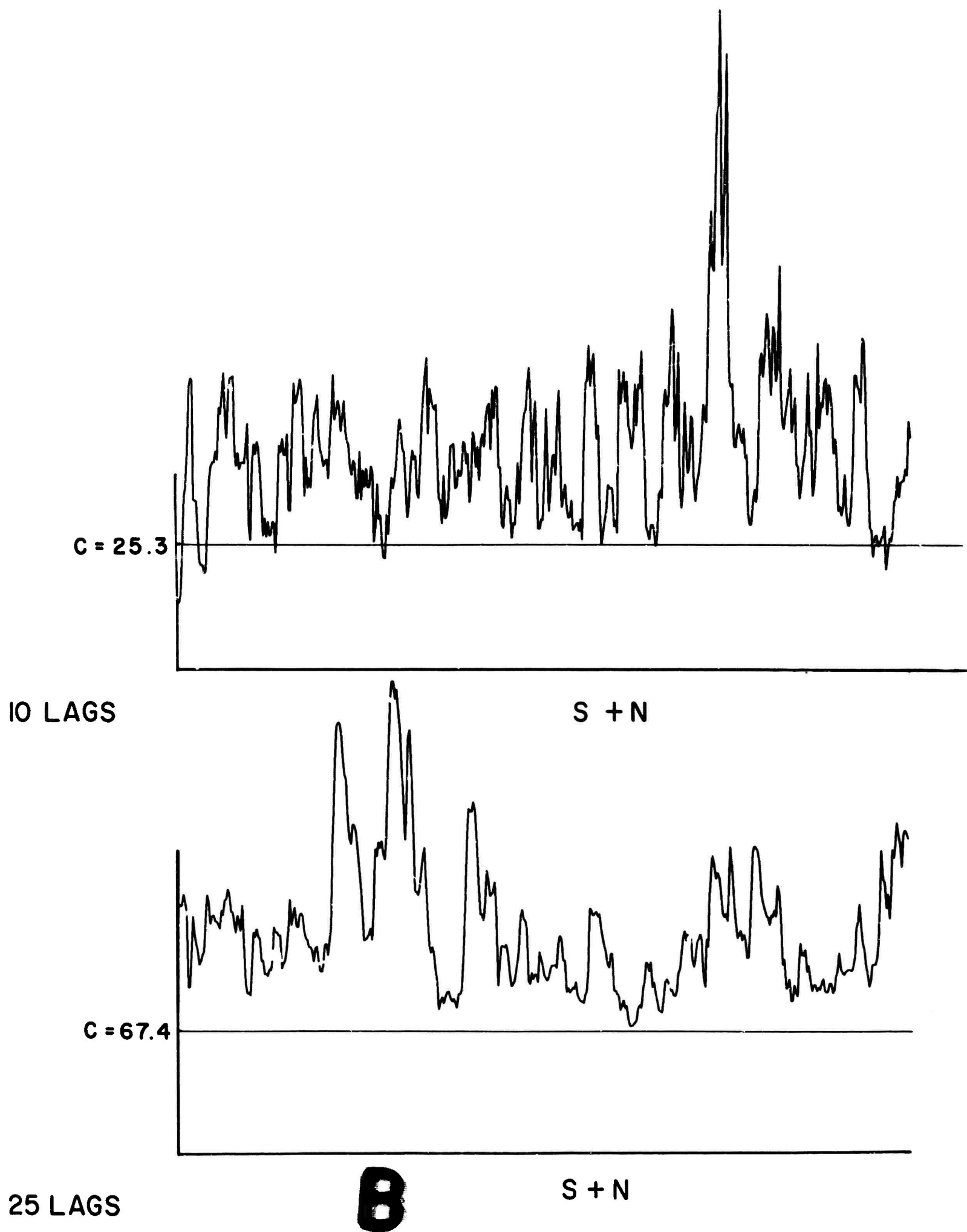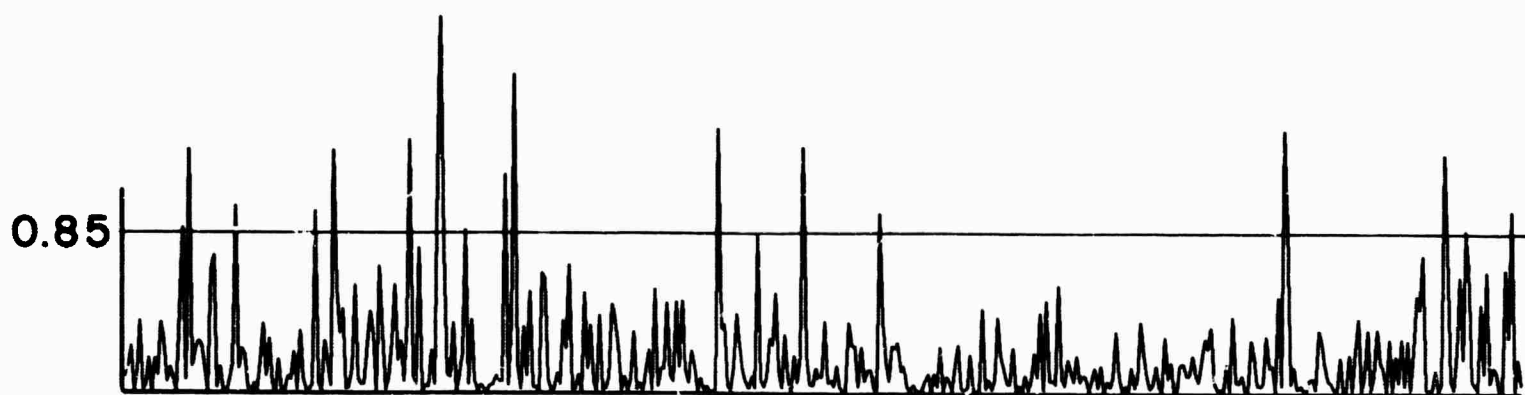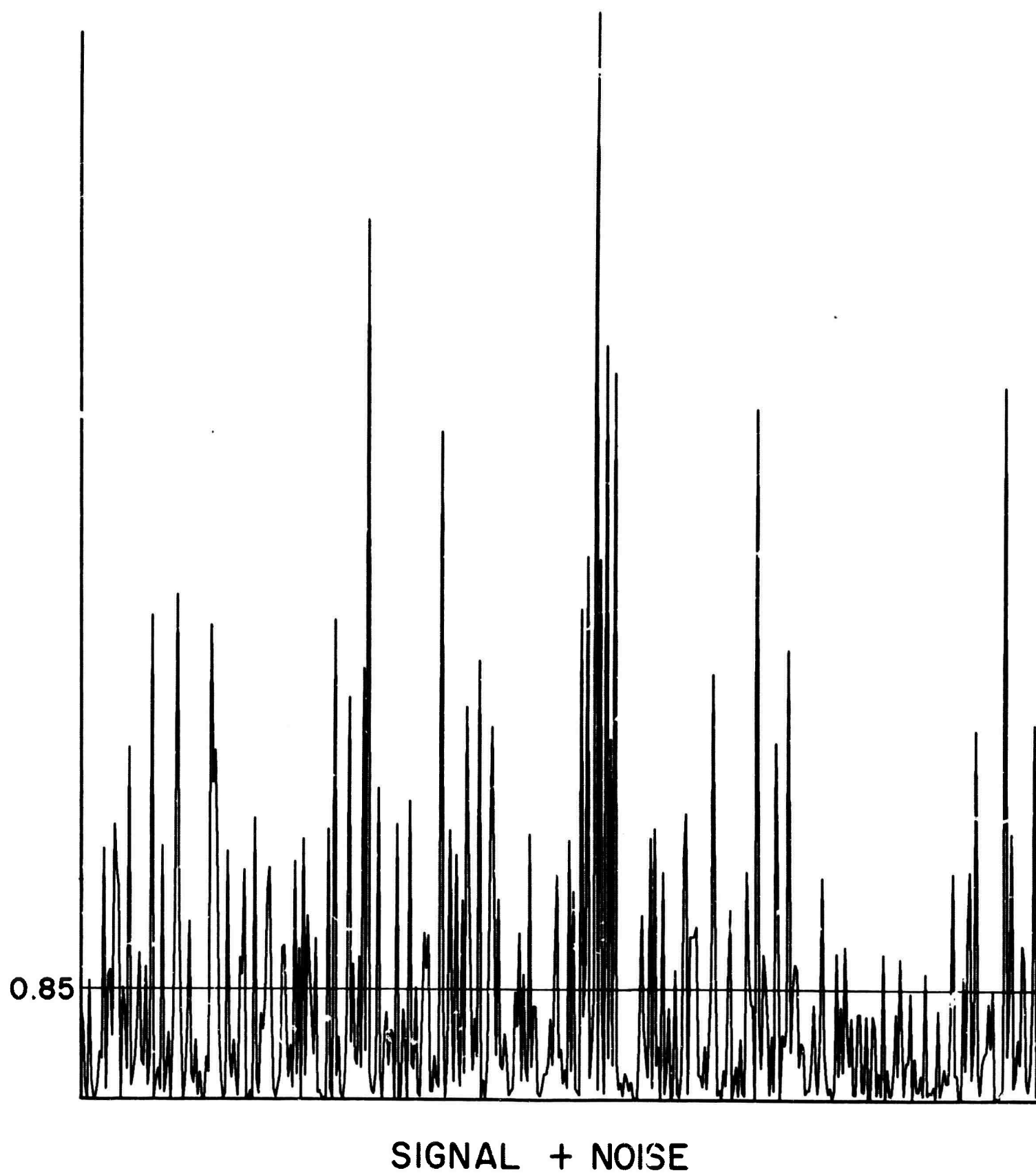
C = 25.3

N   IO

C = 67.4

N

A

25

C = 25.3

10 LAGS   S + N

C = 67.4

25 LAGS   B   S + N
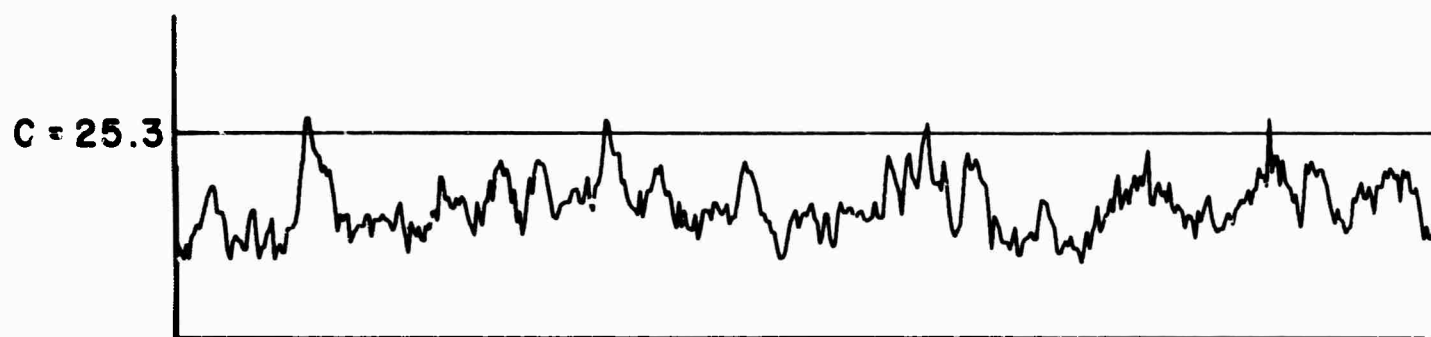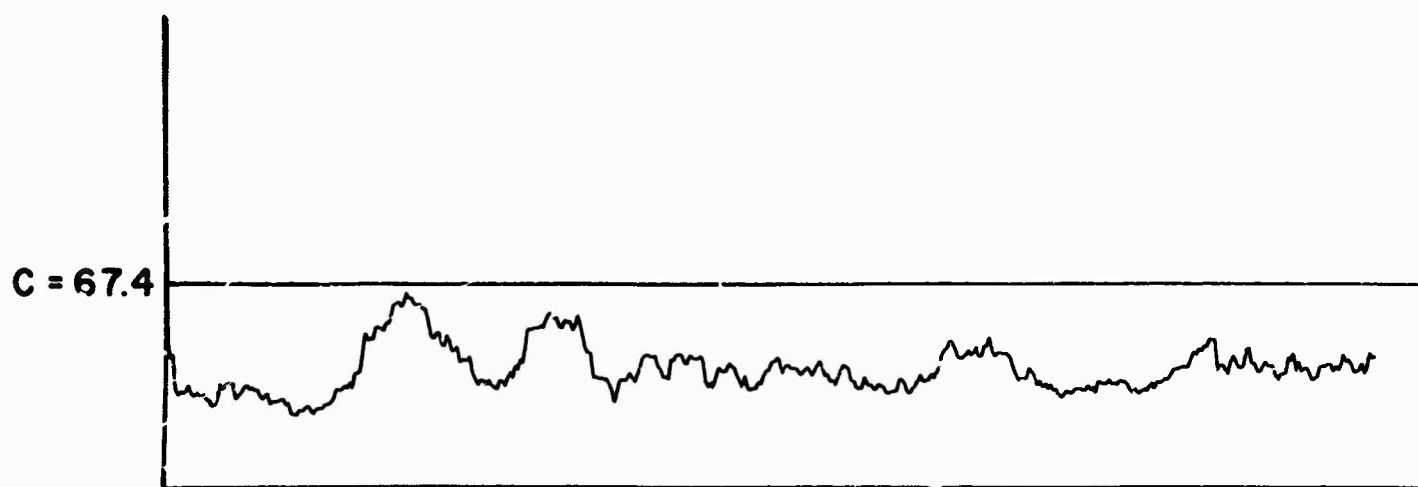
Figure 1. Evaluation of the Quadratic Processor for S = 8 km/sec Solid
Disk Signal, N = 3-4 km/sec Annular Ring Noise, 4 Channels
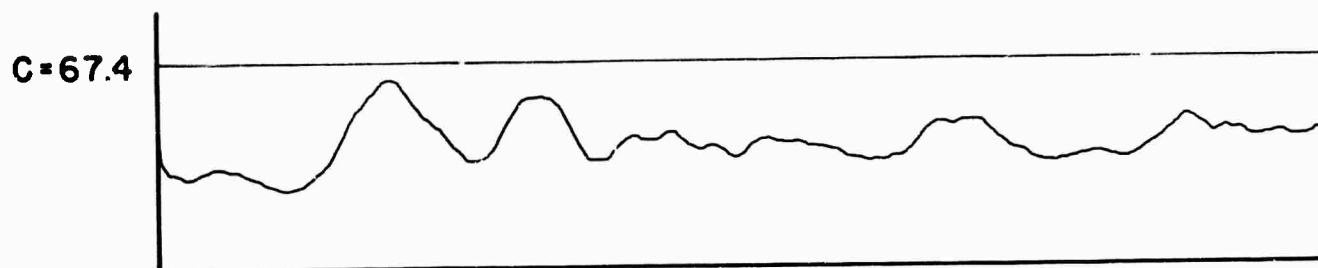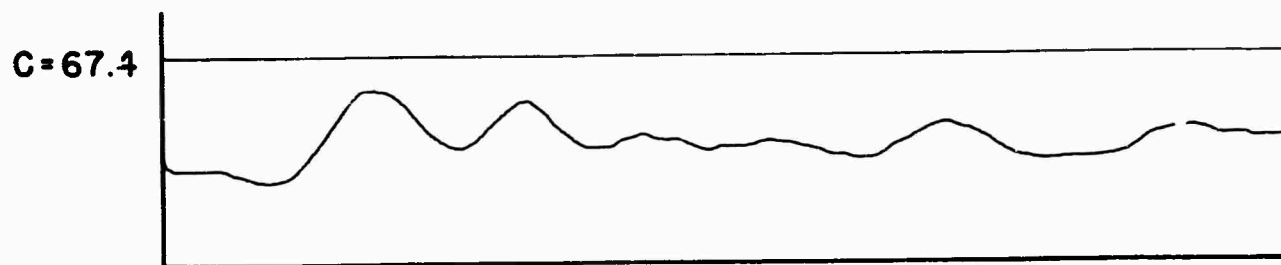(Center and Outer Locations of the WMSO Array)

0.85

NOISE

A

0.85

SIGNAL + NOISE

B

Figure 2. Wiener Filter, Sum and Square Process — 4 Channels, 25 Lags

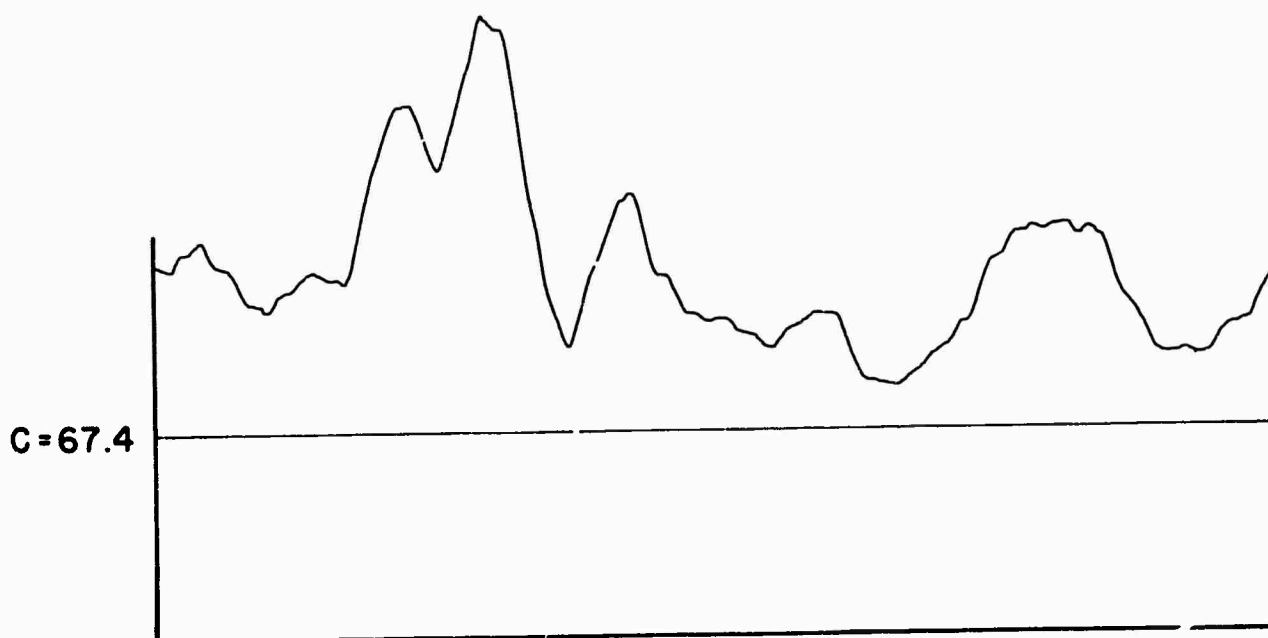C = 67.4

NOISE

SMOOT

C = 67.4

NOISE

SMOOT

A

C = 67.4
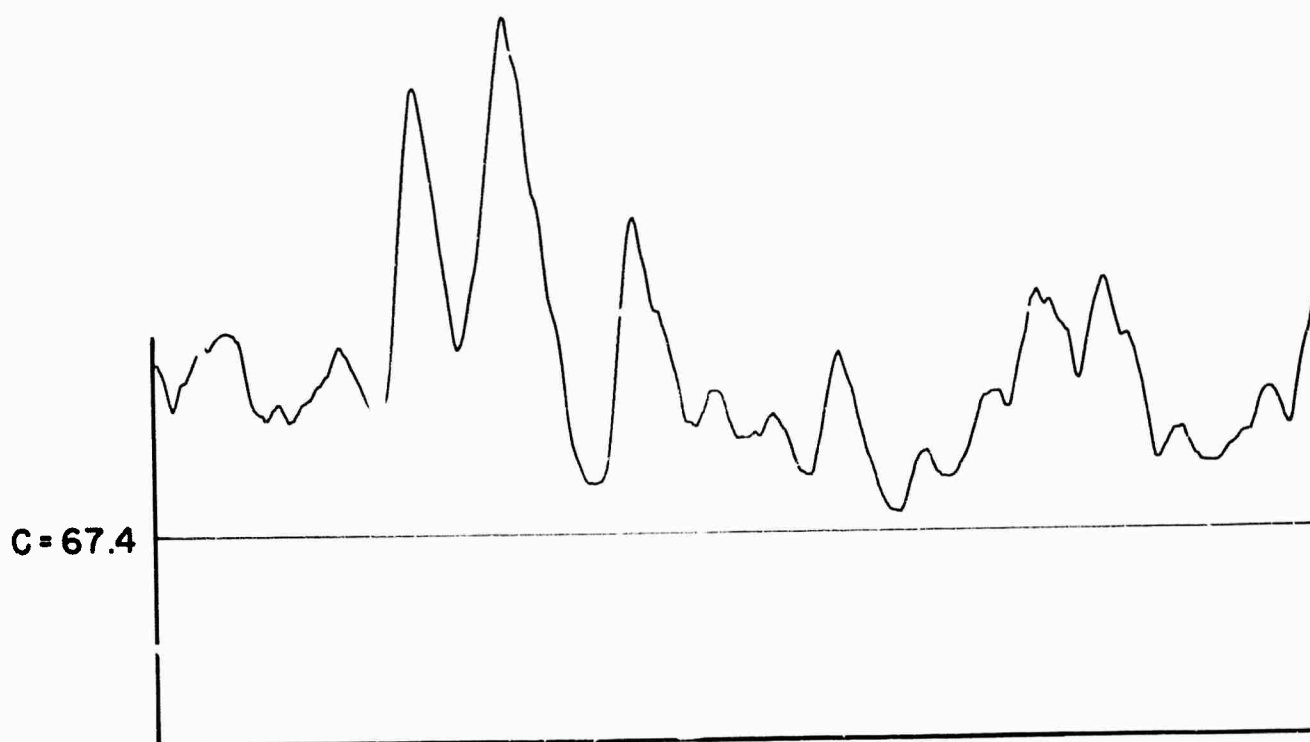
SIGNAL + NOISE

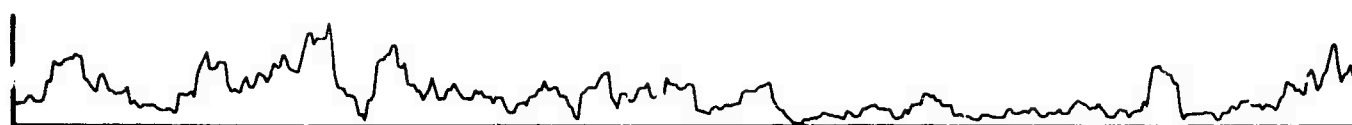SMOOTHED OVER 10 POINTS

C = 67.4

SIGNAL + NOISE

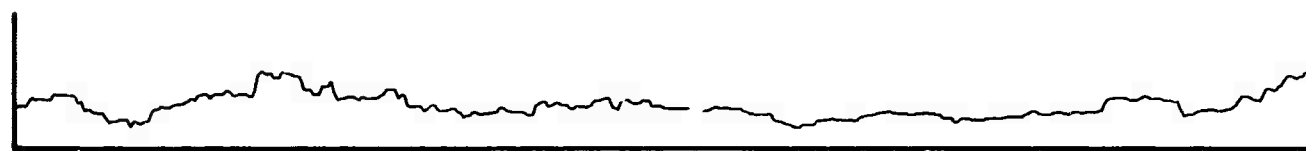SMOOTHED OVER 25 POINTS

B

Figure 3. Results of Figure 1 Smoothed Over 10, 25 Points, Quadr .ic Processor, 25 Lags
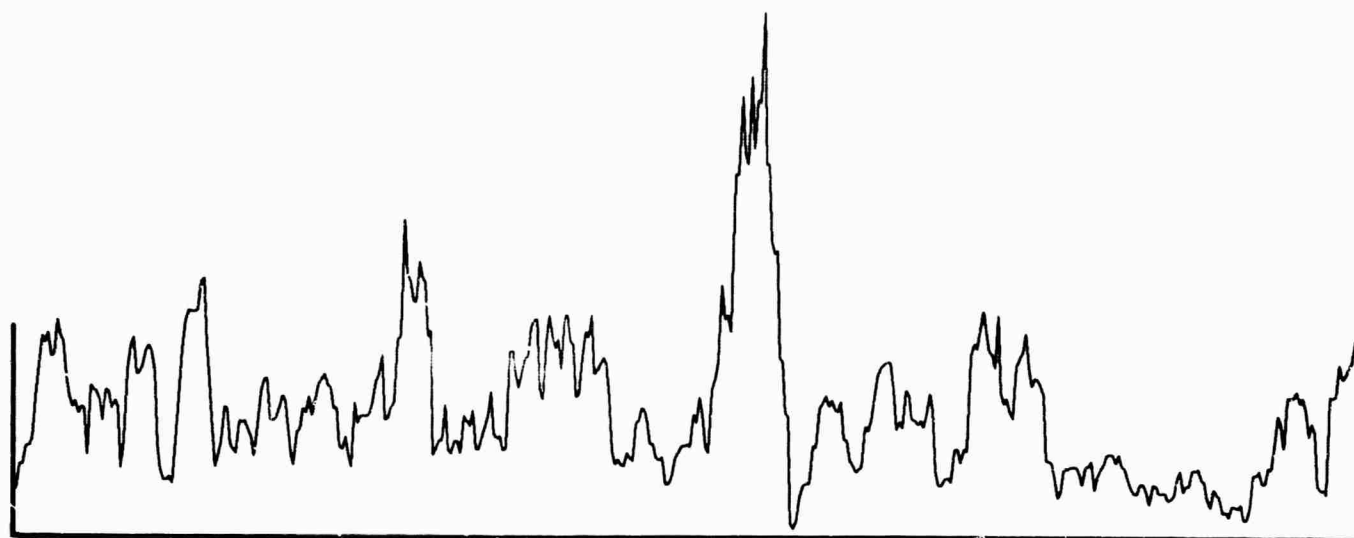
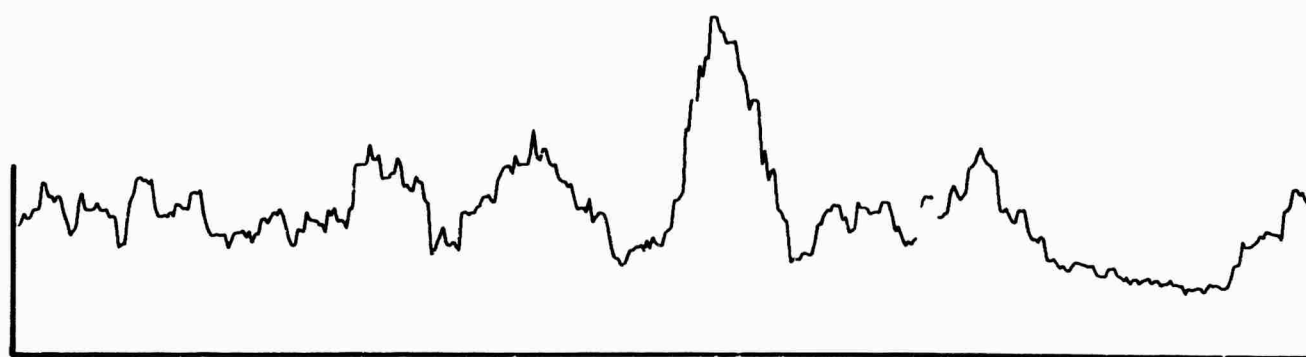NOISE

SMOOTHED OVER

NOISE

SMOOTHED OVER

A

SIGNAL + NOISE

F OVER 10 POINTS

SIGNAL + NOISE

F OVER 25 POINTS

B

Figure 4. Results of Figure 2 Smoothed Over 10, 25 Points,
Wiener Filter Sum and Square Process

Table 1. QUANTITATIVE RESULTS FOR 25 LAGS

	I	II
m	43.58	136.28
s^2	97.73	1570.08
μ	42.76	148.90
σ^2	47.20	1020.17
$|\Omega|$	35.44×10^{-13}	16.84×10^{17}

$$c_{25} = 67.45$$

$$\text{trace } \Omega_2 \, \Omega_1^{-1} = 248.90$$

$$\text{trace } \Omega_1 \, \Omega_2^{-1} = 57.24$$

$$\text{trace } \Omega_2 \, \Omega_1^{-1} \Omega_2 \, \Omega_1^{-1} = 1417.96$$

$$\text{trace } \Omega_1 \, \Omega_2^{-1} \Omega_1 \, \Omega_2^{-1} = 38.08$$

V. References

[1] Array Research Basic Theory of Probabilistic Processing Special Report No. 4, December 3, 1964.

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[3] Siddiqui, M. M., Approximations to the Distribution of Quadratic Forms Annals of Mathematical Statistics, v. 36, pp. 677, April, 1965.

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